3.1 Exercises

1. (a) How is the number e defined?
   (b) Use a calculator to estimate the values of the limits

$$\lim_{h \to 0} \frac{2.7^h - 1}{h} \quad \text{and} \quad \lim_{h \to 0} \frac{2.8^h - 1}{h}$$

   correct to two decimal places. What can you conclude about the value of e?

2. (a) Sketch, by hand, the graph of the function $f(x) = e^x$, paying particular attention to how the graph crosses the y-axis. What fact allows you to do this?
   (b) What types of functions are $f(x) = e^x$ and $g(x) = x^e$? Compare the differentiation formulas for f and g.
   (c) Which of the two functions in part (b) grows more rapidly when x is large?

3–28 □ Differentiate the function.

3. $f(x) = 5x - 1$
4. $F(x) = -4x^{10}$
5. $f(x) = x^2 + 3x - 4$
6. $g(x) = 5x^8 - 2x^5 + 6$
7. $y = x^{-2/5}$
8. $y = 5e^x + 3$
9. $V(r) = \frac{4}{3}\pi r^3$
10. $R(t) = 5t^{-3/5}$
11. $Y(t) = 6t^{-9}$
12. $R(x) = \dfrac{\sqrt{10}}{x^7}$
13. $F(x) = (16x)^3$
14. $y = \sqrt[3]{x}$
15. $g(x) = x^2 + \dfrac{1}{x^2}$
16. $f(t) = \sqrt{t} - \dfrac{1}{\sqrt{t}}$
17. $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}}$
18. $y = \dfrac{x^2 - 2\sqrt{x}}{x}$
19. $y = 3x + 2e^x$
20. $y = \sqrt{x}\,(x - 1)$
21. $y = 4\pi^2$
22. $y = x^{4/3} - x^{2/3}$
23. $y = ax^2 + bx + c$
24. $y = A + \dfrac{B}{x} + \dfrac{C}{x^2}$
25. $y = x + \sqrt[5]{x^2}$
26. $u = \sqrt[3]{t^2} + 2\sqrt{t^3}$
27. $v = x\sqrt{x} + \dfrac{1}{x^2\sqrt{x}}$
28. $y = e^{x+1} + 1$

29–34 □ Find $f'(x)$. Compare the graphs of f and f' and use them to explain why your answer is reasonable.

29. $f(x) = 2x^2 - x^4$
30. $f(x) = 3x^5 - 20x^3 + 50x$
31. $f(x) = 3x^{15} - 5x^3 + 3$
32. $f(x) = x + \dfrac{1}{x}$
33. $f(x) = x - 3x^{1/3}$
34. $f(x) = x^2 + 2e^x$

35. (a) By zooming in on the graph of $f(x) = x^{2/5}$, estimate the value of $f'(2)$.
    (b) Use the Power Rule to find the exact value of $f'(2)$ and compare with your estimate in part (a).

36. (a) By zooming in on the graph of $f(x) = x^2 - 2e^x$, estimate the value of $f'(1)$.
    (b) Find the exact value of $f'(1)$ and compare with your estimate in part (a).

37–40 □ Find an equation of the tangent line to the curve at the given point. Illustrate by graphing the curve and the tangent line on the same screen.

37. $y = x + \dfrac{4}{x}, \quad (2, 4)$
38. $y = x^{5/2}, \quad (4, 32)$
39. $y = x + \sqrt{x}, \quad (1, 2)$
40. $y = x^2 + 2e^x, \quad (0, 2)$

41. (a) Use a graphing calculator or computer to graph the function $f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30$ in the viewing rectangle $[-3, 5]$ by $[-10, 50]$.
    (b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of f'. (See Example 1 in Section 2.9.)
    (c) Calculate $f'(x)$ and use this expression, with a graphing device, to graph f'. Compare with your sketch in part (b).

42. (a) Use a graphing calculator or computer to graph the function $g(x) = e^x - 3x^2$ in the viewing rectangle $[-1, 4]$ by $[-8, 8]$.
    (b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of g'. (See Example 1 in Section 2.9.)
    (c) Calculate $g'(x)$ and use this expression, with a graphing device, to graph g'. Compare with your sketch in part (b).

43. Find the points on the curve $y = x^3 - x^2 - x + 1$ where the tangent is horizontal.

44. For what values of x does the graph of $f(x) = 2x^3 - 3x^2 - 6x + 87$ have a horizontal tangent?

45. Show that the curve $y = 6x^3 + 5x - 3$ has no tangent line with slope 4.

46. At what point on the curve $y = 1 + 2e^x - 3x$ is the tangent line parallel to the line $3x - y = 5$? Illustrate by graphing the curve and both lines.

47. Draw a diagram to show that there are two tangent lines to the parabola $y = x^2$ that pass through the point $(0, -4)$. Find the coordinates of the points where these tangent lines intersect the parabola.

48. Find the equations of both lines through the point $(2, -3)$ that are tangent to the parabola $y = x^2 + x$.

49. The **normal line** to a curve C at a point P is, by definition, the line that passes through P and is perpendicular to the tangent line to C at P. Find an equation of the normal line to the parabola $y = 1 - x^2$ at the point $(2, -3)$. Sketch the parabola and its normal line.

50. Where does the normal line to the parabola $y = x - x^2$ at the point $(1, 0)$ intersect the parabola a second time? Illustrate with a sketch.

51. Use the definition of a derivative to show that if $f(x) = 1/x$, then $f'(x) = -1/x^2$. (This proves the Power Rule for the case $n = -1$.)

52. Find a parabola with equation $y = ax^2 + bx$ whose tangent line at $(1, 1)$ has equation $y = 3x - 2$.

53. Let

$$f(x) = \begin{cases} 2 - x & \text{if } x \leqslant 1 \\ x^2 - 2x + 2 & \text{if } x > 1 \end{cases}$$

Is f differentiable at 1? Sketch the graphs of f and f'.

54. At what numbers is the following function g differentiable?

$$g(x) = \begin{cases} -1 - 2x & \text{if } x < -1 \\ x^2 & \text{if } -1 \leqslant x \leqslant 1 \\ x & \text{if } x > 1 \end{cases}$$

Give a formula for g' and sketch the graphs of g and g'.

55. (a) For what values of x is the function $f(x) = |x^2 - 9|$ differentiable? Find a formula for f'.
(b) Sketch the graphs of f and f'.

56. Where is the function $h(x) = |x - 1| + |x + 2|$ differentiable? Give a formula for h' and sketch the graphs of h and h'.

57. For what values of a and b is the line $2x + y = b$ tangent to the parabola $y = ax^2$ when $x = 2$?

58. Let

$$f(x) = \begin{cases} x^2 & \text{if } x \leqslant 2 \\ mx + b & \text{if } x > 2 \end{cases}$$

Find the values of m and b that make f differentiable everywhere.

59. Find a cubic function $y = ax^3 + bx^2 + cx + d$ whose graph has horizontal tangents at the points $(-2, 6)$ and $(2, 0)$.

60. A tangent line is drawn to the hyperbola $xy = c$ at a point P.
(a) Show that the midpoint of the line segment cut from this tangent line by the coordinate axes is P.
(b) Show that the triangle formed by the tangent line and the coordinate axes always has the same area, no matter where P is located on the hyperbola.

61. Evaluate $\displaystyle\lim_{x \to 1} \frac{x^{1000} - 1}{x - 1}$.

62. Draw a diagram showing two perpendicular lines that intersect on the y-axis and are both tangent to the parabola $y = x^2$. Where do these lines intersect?

3.2 The Product and Quotient Rules

The formulas of this section enable us to differentiate new functions formed from old functions by multiplication or division.

The Product Rule

By analogy with the Sum and Difference Rules, one might be tempted to guess, as Leibniz did three centuries ago, that the derivative of a product is the product of the derivatives. We can see, however, that this guess is wrong by looking at a particular example. Let $f(x) = x$ and $g(x) = x^2$. Then the Power Rule gives $f'(x) = 1$ and $g'(x) = 2x$. But $(fg)(x) = x^3$, so $(fg)'(x) = 3x^2$. Thus, $(fg)' \neq f'g'$. The correct formula was discovered by Leibniz (soon after his false start) and is called the Product Rule.

Before stating the Product Rule, let's see how we might discover it. In the case where $u = f(x)$ and $v = g(x)$ are both positive functions, we can interpret the product uv as an area of a rectangle (see Figure 1). If x changes by an amount Δx, then the corresponding changes in u and v are

$$\Delta u = f(x + \Delta x) - f(x) \qquad \Delta v = g(x + \Delta x) - g(x)$$

and the new value of the product, $(u + \Delta u)(v + \Delta v)$, can be interpreted as the area of the large rectangle in Figure 1 (provided that Δu and Δv happen to be positive).

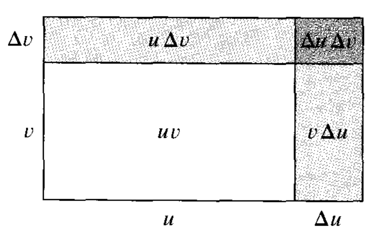

FIGURE 1
The geometry of the Product Rule